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Velocity-dependent mass or proper time?

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Abstract. Physics teachers and high-energy physicists express different opinions concerning the velocity-dependent mass. To understand this issue better a résumé of velocity-dependent mass as well as of proper time is given in a historical setting. The role of both concepts in physics teaching is considered. From a four-dimensional point of view in elementary special relativity the introduction of proper time appears to be promising. The advantages and shortcomings of the replacement of coordinate time with proper time and of constant mass with velocity-dependent mass are discussed.

Zusammenfassung. Physiklehrer und Hochenergiephysiker vertreten bezüglich der geschwindigkeitsabhängigen Masse verschiedene Ansichten. Um den Gegensatz besser zu verstehen wird eine geschichtliche Übersicht über die geschwindigkeitsabhängige Masse sowohl als auch über die Eigenzeit gegeben. Die Rolle beider Begriffe im Physikunterricht wird erörtert. Vom vierdimensionalen Standpunkt scheint die Einführung der Eigenzeit in der elementaren speziellen Relativitätstheorie viel zu versprechen. Die Vor- und Nachteile der Ersetzung der Koordinatenzeit mit der Eigenzeit und der Masse mit der geschwindigkeitsabhängigen Masse werden diskutiert.

1. Introduction

The special theory of relativity is taught, almost everywhere, at secondary school level as a welcome example of a theory that transcends everyday experience. In this article first velocity-dependent mass, being an older concept than special relativity, the energy–mass relation and proper time are considered in a historical setting. The replacement of Newtonian mass with velocity-dependent mass as a pedagogical vehicle is reviewed. After briefly considering a four-dimensional approach, another pedagogical vehicle, the replacement of absolute time with proper time, is presented and its advantages and shortcomings are discussed.

2. A survey

The idea of *velocity-dependent mass* actually sprang *out of the problem of non-integral atomic masses* (Siegel 1978). J J Thomson studied the motion of a charged hollow sphere and introduced an effective mass made up of the bare mass and the electromagnetic mass (Whittaker 1953, Pais 1982). Thereby he followed G G Stokes who considered the hydrodynamical effective mass of a sphere moving in an ideal fluid. Thomson's expression for the electromagnetic mass was corrected to $\frac{4}{3}E/c^2$ with the electrostatic energy E by Heaviside in 1889. M Abraham recognized that this was an approximation for small velocities and developed a more general expression. A H Lorentz (1904), A Bucherer (1904), P Langevin (1905), H Poincaré (1905) and others considered still more complicated models. Lorentz used the longitudinal mass $\gamma^3 m$ and the transversal mass γm with $\gamma = (1 - v^2/c^2)^{-1/2}$ for electrons, assuming that they experience the FitzGerald contraction; the main idea he had already put forward in 1899. In 1900 Poincaré gave the relation E/c^2 for the mass associated with a free electromagnetic field with energy E. He considered the momentum density to be equal to the energy flux density divided by c^2. In 1904 he spoke of inertia increasing with velocity and considered the velocity of light as an upper limit.

In 1905 came Einstein's article containing all the results of special relativity concerning space, time and particle motion with the longitudinal and transversal mass and the kinetic energy (Einstein 1905a). It is interesting to note that the transversal mass was given as $\gamma^2 m$, although the equation for magnetic deflection was correct. The expression for the transversal mass was corrected by M Planck in 1906 who also studied the momentum $m\gamma\boldsymbol{v}$ of a particle. G N Lewis in 1908 proved the mass–energy relation considering the radiation pressure on a body which is absorbing energy and distinguished total energy $mc^2\gamma$ and rest energy mc^2. In the following year together with R Tolman he showed that in an elastic collision of two particles the momentum is conserved. Tolman in 1912

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advocated the use of relativistic mass $m\gamma$. W Pauli in 1921 explicitly abandoned the longitudinal and transversal mass but retained the relativistic mass.

The *mass–energy relation* was derived within special relativity in a short article by Einstein (1905b). He considered a body emitting two equal electromagnetic wave packets in opposite directions. The conservation of total energy was exploited in the rest frame of the body and in a moving frame of reference whereby the transformation of the energy of the electromagnetic wave packet was taken as known. Subtracting the equation for the rest frame from the equation for the moving frame and taking into account the expression for the kinetic energy of the body the relation $(m_i - m_f)c^2 = 2E_1$ with the energy of the wave packet in the rest frame E_1 is obtained. Einstein had developed, in his first article, all the equations needed but he preferred to take the approximation for a small relative velocity of both frames so that the Newtonian kinetic energy could be used†.

This derivation was criticized by H Ives in 1952 and many other physicists agreed, e.g. M Jammer (1961). Only recently J Stachel and R Torretti (1982) refuted the critique. According to W L Fadner (1988), who considered the historical development in detail, their conclusion is sound though their way to it may not be irreproachable.

M J Feigenbaum and N D Mermin (1988) derived the relation for a body of mass m_i which emits two particles with equal mass m_1 and equal velocity in opposite directions. With the equation for the total energy of a particle and its transformation, without further suppositions, the result is obtained that the kinetic energy of both particles in the rest frame of the body is equal to $m_ic^2 - (m_f + 2m_1)c^2$. F Rohrlich (1990) simplified Einstein's derivation so that only the equation for the linear Doppler effect is needed.

C G Adler (1987) discussed the relativistic mass in a pedagogical and historical context and called attention to the fact that various introductory textbooks, as well as various editions of the same text, can show a different attitude towards the concept of relativistic mass. This may trouble students and even teachers to an appreciable extent. Adler considered in detail, including the approximations of general relativity, the notion that the relativistic mass describes the inertia of a body and found it inappropriate‡.

† In the following year Einstein studied a hollow cylinder which emitted an electromagnetic wave packet from one end and absorbed it at the other. The centre-of-mass of an independent system should stay at rest so the momentum of the packet E_1/c equals the mass, corresponding to the packet, multiplied by c. According to Maxwell's electrodynamics a packet with energy E_1 has a momentum E_1/c.

‡ The difference between the longitudinal and transversal mass makes the notion of the relativistic mass as a measure of inertia doubtful. The proportionality of force and acceleration can be expressed by a mass tensor (Rockower 1987).

L B Okun (1989a, b) called attention to the fact that in high-energy physics the relativistic mass is not used, and took a stand against obsolete notions in the teaching of special relativity. A particle has only one mass m and there is no reason to introduce the relativistic mass m_r and the rest mass m_0 if consistent terminology and symbols are to be used. The equation $E_0 = mc^2$ is well founded, whereas $E = mc^2$ is not. Okun's proposition was not met with unequivocal approval. W Rindler (1990), for example advocated the use of relativistic mass.

The concept of *proper time* has its roots in the time dilation which can be deduced from the Lorentz transformation. W Voigt, who derived the transformation up to a common scale factor in 1887, was not aware of an effect symmetric to the FitzGerald contraction. As he did not realize that the wave equation is invariant against his transformation he was not 'a premature discoverer of the Lorentz transformation' (Kittel 1974, Doyle 1988). J Larmor who derived the Lorentz transformation in its present form in two steps in 1900 gave, in 1887, 'the first historical statement of time dilation'. However, he did not, as his later remarks show, see in it an expression of the relativity principle. Also Lorentz who independently derived the transformation, first up to terms linear and then quadratic in v/c and 1904 in its final form, did not grasp the full physical content of time dilation. He persisted in considering the universal time as measured in the ether as the true time and the local time in the moving frame as a mathematical convenience. So Einstein, deriving the Lorentz transformation independently and abandoning the ether, was the first to consider time dilation in the present context (Einstein 1905a, Rindler 1970). Proper time τ was introduced by $d\tau = dt/\gamma$ as an invariant in four-dimensional spacetime in 1908 by H Minkowski in an appendix to his seminal paper (Whittaker 1953).

3. Traditional elementary approaches

There is a multitude of introductory textbooks of special relativity which differ in both rigour and breadth (Dorling 1979). Nevertheless, velocity-dependent mass appears to be the central part of dynamics in most traditional introductory textbooks. Often, at or near the beginning, the Newtonian mass is replaced with the *relativistic* or *dynamical mass*

$$m \rightarrow m_r = \frac{m}{\sqrt{1 - v^2/c^2}} = \gamma m \qquad (1)$$

realizing that the relativistic momentum $m\gamma\boldsymbol{v}$ is obtained, using (1), from the Newtonian momentum $m\boldsymbol{v}$. Then the equation of motion

$$\boldsymbol{K} = \frac{d\boldsymbol{P}}{dt} = \frac{d(m_r\boldsymbol{v})}{dt} = m\frac{d(\gamma\boldsymbol{v})}{dt} = q(\mathscr{E} + \boldsymbol{v} \times \mathscr{B}) \qquad (2a)$$

is obtained from Newton's law by way of (1) equating $\boldsymbol{K}$ with the Lorentz force. $\mathscr{E}$ is the electric and $\mathscr{B}$ the magnetic field. For longitudinal electric or transversal magnetic field the equation of motion becomes

$$\gamma^3 m \frac{\mathrm{d}v}{\mathrm{d}t} = q\mathscr{E}$$

or (2b)

$$\gamma m \frac{\mathrm{d}v}{\mathrm{d}t} = m\gamma \frac{v^2}{r_0} = qv\mathscr{B}$$

respectively, where $r_0 = m\gamma v/q\mathscr{B}$ is the radius of curvature of the trajectory.

The work done, i.e. the integral from 0 to v of the force $\boldsymbol{K}$ multiplied by the displacement $\boldsymbol{v}\,\mathrm{d}t$, gives the kinetic energy of the body

$$T = mc^2\gamma - mc^2 = m_r c^2 - mc^2 \qquad (3)$$

as the mass increment multiplied by c^2. The conclusion that the total energy is made up of the kinetic and the rest energy, $m_r c^2 = T + mc^2$, is generalized to other forms of energy and used to study reactions and decays of composite particles.

4. A four-dimensional approach

In particle physics special relativity is used in its four-dimensional form. A simple version adapted to sophomores can be based on two assumptions (see also Strnad 1984).

(i) *Lorentz covariance*. Quantities of physics are identified with elements of the pseudo-Euclidic four-dimensional world: scalars (invariants), 4-vectors, tensors. Laws of physics are expressed by covariant equations containing these quantities.

(ii) *Correspondence*. At velocities small compared with c, the velocity of light, the covariant equations go over into classical ones.

The time and the position of an event are specified by the world 4-vector $(ct, \boldsymbol{r})$ whose scalar product with itself is an invariant, $c^2t^2 - \boldsymbol{r}\cdot\boldsymbol{r} = c^2\tau^2$. Thereby τ is the invariant proper time of an event with respect to $(0, 0, 0, 0)$. From the corresponding equation for two adjacent events the relation between the coordinate time interval $\mathrm{d}t$ and the proper time interval $\mathrm{d}\tau$ is obtained as $\mathrm{d}\tau = \mathrm{d}t/\gamma$.

The 4-velocity is introduced as the derivation of the world vector with respect to the proper time, and the 4-momentum by multiplying the latter with mass. The mass of a particle that preserves its identity, called a point particle for short, is invariant. The equation of motion states that the derivation of 4-momentum with respect to proper time is equal to the 4-force. For a charged particle in the electric and magnetic field the 4-force is made up of the charge q which is a scalar, the 4-velocity and the skew symmetric field tensor:

$$\left(m\frac{\mathrm{d}^2(ct)}{\mathrm{d}\tau^2}, m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}\tau^2}\right) = \left(\frac{\mathrm{d}E}{c\,\mathrm{d}\tau}, \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}\tau}\right) = \left(\gamma q\frac{\mathscr{E}\cdot\boldsymbol{v}}{c}, \gamma q(\mathscr{E} + \boldsymbol{v}\times\mathscr{B})\right). \qquad (4)$$

After the motion of a particle, the reactions and decays of composite particles are studied. A system of interacting composite particles, not disturbed by other particles, can be taken as independent. For such a system, among other quantities, the 4-momentum is conserved. The mass of a composite particle in the ground state is given by the rest energy in the centre-of-mass frame, which is equal to the minimum total energy of all constituent particles. The mass is not conserved. The mass difference, i.e. the mass of the binding energy, can be measured directly if the interaction is strong enough. The rest energy of composite particles can be considered to be an energy reservoir. However, other conservation laws, particularly barion and lepton number conservation, limit the changes of rest energy.

5. Proper time in elementary special relativity

On the one hand the four-dimensional form of special relativity is not suited for secondary school and on the other, one would rather not base teaching on notions abandoned in research long ago. As in physics teaching formal statements are unsatisfactory, one is seeking a substitute for the notion that the inertia of a body increases with velocity, and for the replacement (1). A simple possibility is envisaged: the coordinate time, measured by synchronized clocks placed in an inertial frame of reference at points traversed by the particle, is to be replaced by the proper time, measured by the comoving clock

$$\frac{1}{\mathrm{d}t} \rightarrow \frac{1}{\mathrm{d}\tau} = \gamma\frac{1}{\mathrm{d}t}. \qquad (5)$$

The relation between proper and coordinate time can be obtained by considering a light clock in the proper and in a moving inertial reference frame. Thereby the light velocity *in vacuo* which since 1983 is implicitly given by the definition of the metre, i.e. a convention valid in every inertial frame of reference, is taken as frame independent. Measurements of decay time of particles in flight support the relation.

The two assumptions of section 4 are abandoned for (5) and the assumption that one has to start with Newtonian mechanics. Equation (5) can be intuitively easily grasped: time is to be measured by a clock moving with the particle. While the velocity of a particle is small compared to the velocity of light it does not matter whether the clock is moving together with the observed particle or not, but for greater velocities this becomes important. In this way the four-dimensional form of special relativity is not introduced but the stage is set to continue with it later.

One difficulty, however, should not remain unmentioned. In introducing the equation of motion one has to explain separately that on the right-hand side of equation (4) the additional coefficient γ must appear in order that this side transforms as a 4-vector. It is simplest to show that the transverse components of $q\gamma(\mathscr{E} + \boldsymbol{v}g \times \mathscr{B})$ for $\boldsymbol{v} = (v, 0, 0)$ are not changed in going over to another reference frame. This can only be done if the transformations for the field are provided which, in general, is not the case. So one has to refer to measurements with charged particles giving support to equation (2)†.

6. Discussion

As we have seen, special relativity can be introduced by replacing coordinate time with proper time. Thereby each quantity has a precise definition and is represented by a unique symbol. To the author a consequent such approach is not known though it may be considered to be in the spirit of Einstein who characterized the essence of special relativity, very briefly, by the changed notion of time.

Velocity-dependent mass was a fruitful concept, particularly for Einstein in the development of general relativity. After he discovered the equivalence principle and after Minkowski introduced four-dimensional spacetime in 1908 the concept lost its significance. In teaching special relativity, velocity-dependent mass may be popular as it gives an impression of a straightforward derivation of the equation of motion (2). The notion that the inertia is increasing with velocity (1) appears to be intuitively easily accepted but is misleading.

The characteristic of the motion of a particle, particularly the difference with respect to Newtonian mechanics, can be deduced from the total energy $E = mc^2\gamma$ without referring to the relativistic mass. The relativistic energy increment, being singular at $v/c \to 1$, is greater than the corresponding Newtonian one, following from the first for $v/c \to 0$:

$$\mathrm{d}E = mv(1 - v^2/c^2)^{-3/2}\,\mathrm{d}v > mv\,\mathrm{d}v.$$

Conversely, the relativistic velocity increment is smaller than the corresponding Newtonian one, following from the first for $E/mc^2 \to 1$:

$$\mathrm{d}v = m^2c^5E^{-2}(E^2 - m^2c^4)^{-1/2}\,\mathrm{d}E$$
$$< (2m(E - mc^2))^{-1/2}\,\mathrm{d}E.$$

The four-dimensional derivation also shows that the coefficient γ is more of kinematical than of dynamical origin and thus (5) from this standpoint is more pleasing than (1)‡.

† This difficulty does not exist in general relativity for radial motion in the Schwarzschild geometry where the equation of motion can be written in the simple form $m\mathrm{d}^2r/\mathrm{d}\tau^2 = -Gmm_s/r^2$ with the gravitational constant G and the mass m_s of the central body, r being the Schwarzschild radial coordinate.

‡ Often students are led astray by the relativistic mass, introducing, e.g., the relativistic kinetic energy as $\frac{1}{2}mv^2 \to \frac{1}{2}m\gamma v^2$.

It is important to distinguish in this discussion between a point particle and a system of particles. Such a system should be considered after the equations for a particle are given, e.g. in the sense of Feigenbaum and Mermin (1988), as an example for the changing mass and rest energy. Einstein's derivations and Rohrlich's elementary modification are not as suitable as they use electromagnetic wave packets with mass and rest energy equal to zero. Some misunderstandings may arise from not stating clearly whether a particle or system of particles is being considered, whether 'mere translational kinetic energy of a body as a whole' or the internal kinetic energy of constituents of a system of particles contributing to its total energy and mass are considered. The traditional approach obscures this point by exploiting equation (3) directly for systems of particles.

Having quoted the advantages of the approach via proper time we have to take into account that most pupils of secondary school are not physicists and only a few of them will take physics degrees. From their viewpoint the situation may appear different. According to an inquiry among students who intend to major in physics, the idea of velocity-dependent mass seemed strange to them in secondary schools as was the proper time. They have heard vaguely of the time being 'the fourth dimension' but they could not possibly see the usefulness and elegance of the four-dimensional spacetime. Besides, a multitude of traditional textbooks and popular scientific texts rely on the relativistic mass and the 'famous equation' $E = mc^2$. So we are led to the convinction that in physics teaching extreme views are not fruitful, the changes being much slower than in research. A teacher can choose to introduce the 'relativistic mass' (see, e.g., Baird 1980, Tsai 1986) if he or she emphasizes that this is a didactic quirk with a limited domain of validity, not directly connected to inertia, and avoids misunderstandings by, for example, designating the velocity-dependent mass by m_r or even better by $m\gamma$ (and not by m and not introducing the rest mass m_0). Though this approach does not give the whole insight it is less time consuming and needs less background.

Useful quotations

In physics teacher training the following way has been successful. The issue is stated and the pros and cons presented to start the discussion. Quotations from the literature, particularly conflicting ones, are welcome. They are analysed and it is tried to reconcile them mutually by realizing what was actually meant. In this way future teachers are encouraged to choose freely that part of their view that cannot be falsified. Some quotations used are given below.

Es ist nicht gut von der Masse $M = m/\sqrt{1 - v^2/c^2}$ eines bewegten Körpers zu sprechen, da für M keine klare Definition gegeben werden kann. Man beschränkt sich besser auf die "Ruhe-Masse" m. Daneben kann man ja den Ausdruck für Impuls und Energie geben, wenn mann das Trägheitsverhalten rasch bewegter Körper angeben will.

A Einstein in a letter to L Barnett, 19 June 1948 (Okun 1989)

The relationship $E = mc^2$ in which $m = m_0/\sqrt{1 - v^2/c^2}$ is by way of contrast, merely a definition of m and should not be put on the same level of importance as $E_0 = m_0 c^2$.

C G Adler (1987)

In contrast to the remarkable relation ... obeyed by the kinetic energy coefficient, this other use of $E = mc^2$, though it is sometimes cited with comparable fanfare, has very little content.

M J Feigenbaum and N D Mermin (1988)

The concept of rest energy, that is $E_0 = m_0 c^2$, was in many ways the most revolutionary result of the mass–energy relationship.

W L Fadner (1988)

Every year millions of boys and girls throughout the world are taught special relativity in such a way that they miss the essence of the subject. Archaic and confusing notions are hammered into their heads. It is our duty—the duty of professional physicists—to stop this process.

V B Okun (1989a)

Nevertheless, if I am told by the particle physicists —and they *are* the largest user group of special relativity these days—that henceforth I must use the symbol m for rest mass and call it mass, so be it. But I refuse to stop using the concept of relativistic mass, which I would then denote by m_r.

W Rindler (1990)

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